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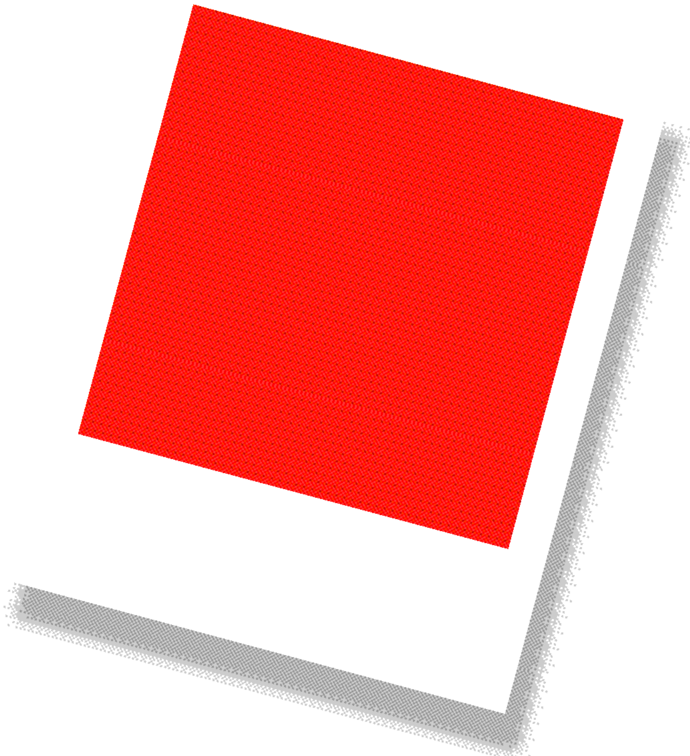
Monopoly 101

The board game can teach us about a lot more than just the untrustworthiness of friends and family. Read well, or else do not pass go, and do not collect $100…



The board game can teach us about lot more than just the untrustworthiness of friends and family. Read well, or else do not pass go, and do not collect $100…Since its introduction in 1935, 500 million people throughout the world have played the board game of Monopoly. Undoubtedly, most of them played casually or badly or both. But some fraction internalized the subtle financial lessons the game teaches. The popularity of the game makes it worthwhile to consider what those lessons are because they could have a profound effect on real financial decisions.

You need not know anything about the game for this article. I will explain the necessary rules as I go along. However, if you do know how to

THE OBJECT AND BASIC RULES

Monopoly is played by two to ten players, four is the ideal number. There is another entity, the Bank, which pays money to and receives money from players in certain circumstances, but does not participate actively in the game. Usually one player is designated to act for the Bank, but in large or serious games a non-playing banker is desirable. If a player is Banker, she merely performs the administrative functions of the Bank; it has no effect on her game play.

The Monopoly board has 40 squares arranged around the outside. The game begins with the Bank handing out $1,500 to each player. Players select tokens and place them on the square marked "Go." Players take turns rolling two six-sided dice, and moving their tokens the number of squares indicated. The game continues until all players but one have gone bankrupt.

The US rules are blunt about this: "In monopoly each player tries to invest 1,500 script dollars to such good advantage that all other players are forced out of the game. A player who has lost his money is bankrupt and leaves the game." One player ends up rich and everyone else leaves the game broke. This causes criticism of the game, a more sensible economic goal might be to cooperate so everyone gets what they reasonably want. This is a serious point but we cannot address it properly until we do a little math.

28 of the 40 squares are "properties." These are the most important feature of the game. The first player whose token lands on a property may buy it from the Bank at a set price marked on the property deed If he declines to buy, the Bank immediately auctions the property to the highest bidder. If a player's token lands on a property already owned by another player, she must pay the owner a set rent, also marked on the property.

play, and you probably do, you will have to bear with my simplifications. Trust that I haven't forgotten about the details, I will get to all of them eventually. Our strategy, like the modern finance pioneers, is to simplify things to the point that we can get an analytic result and corresponding insight. Then we will use that result and insight to build more complex models until we know how to play Monopoly. If you think you know how to play Monopoly already, I think you're wrong.

The key to Monopoly is property valuation. It is essential to know when to buy a property if you land on it, how much to bid at auction and how to buy from and trade with other players to assemble the portfolio of properties you need to win. In casual games, most players buy every property they land on, auctions are rare and a burst of suboptimal trading occurs only once in the middle of the game. In well-played games auctions are frequent and there is a continuous, active market in buying and trading properties.

Clearly a property's value derives from the rent you expect to collect from other players by owning it. We are going to start by ignoring randomness and considering only the long-run averages. We make two further simplifying assumptions:

- All squares are visited with the same long-run frequency
- All players survive until the last property is purchased from the Bank

At that point in the game, each player will have a rent roll which we denote $\Re_i$ for i = 1 to n (the number of players in the game). This is the sum of the rents on the property player i owns, divided by 40. Player i will collect this amount, on average, every time an opponent rolls the dice. Player i will pay, on average,

$$\sum_{j=1,j\neq i}^{n} \Re,$$

every time he rolls the dice.

Now we have to consider the 12 non-property squares. Two of them require the player to pay money to the Bank, $200 in one case and $75 in the other. One of them ("Go") requires the Bank to pay the player $200 every time she lands on or passes it. Six of them require the player to draw a card (if you're counting two of the remaining three squares have no effect, one sends a player to a different square). There are 32 cards and the net amount of the payments to or from the Bank is $485 in the player's favor. Therefore, every time a player rolls the dice, she expects to collect

$$\frac{1}{40} \times \left[-\$200 - \$75 + 7 \times \$200 + 6 \times \frac{\$485}{32}\right] = \$30.40.$$

We will denote this amount as Φ.Most casual players modify the rules to increase Φ.Two common ways to do that are to pay $300 for landing on Go and pay $500, plus any money paid to the bank for landing on squares or drawing cards, in the middle of the board and awarding it to players landing on the "Free Parking" space. This will increase Φ to

$$\frac{1}{40} \times \left[7 \times \$200 + 6 \times \frac{\$800}{32} + \$100 + \$500\right] = \$53.75.$$

That has a profound influence on the game. Among other things it lengthens the game considerably and eliminates any advantage to intelligent play.

Combining property rents to and from other players with money received from the Bank, on each complete round of turns, player i expects to collect a total of

$$\Phi + (n-1)\Re_i - \sum_{j=1,j\neq i}^{n} \Re, = \Phi + n\Re_i - \sum_{j=1}^{n} \Re,.$$

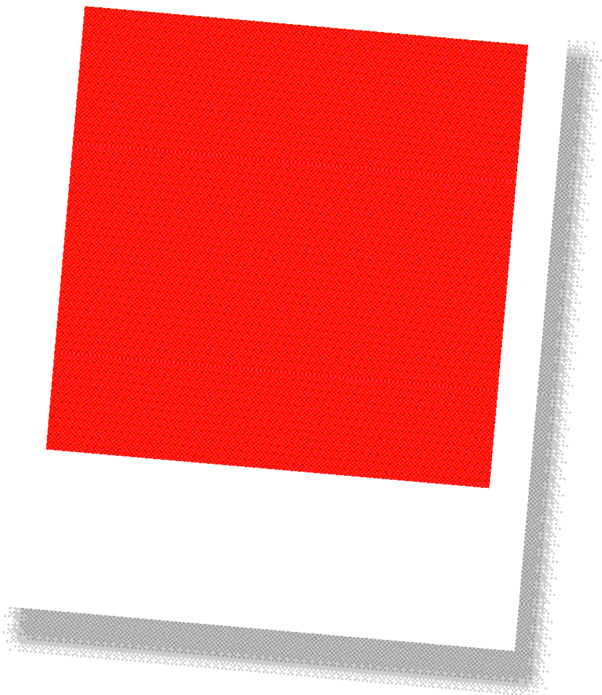
If this is less than zero, player i should eventually go bankrupt. That will happen if player i's rent roll is less than the average player's rent roll minus $\frac{\Phi}{n}$. If no player has a rent roll this much below the average, the game should continue forever, with no winner and no losers.

The total rent of all 28 properties on the board is $547, the total rent roll is $\frac{\$547}{40} = \$13.68$.In a four-person game with all properties owned, the average rent roll is $\frac{\$13.68}{4} = \$3.42$Since $\frac{\Phi}{n} = \frac{\$30.40}{4} = \$7.60$, even a player with no properties (and therefore a zero rent roll) will stay in the game forever. All properties are worthless, but you can pay any amount you like for them, because whatever happens no one ever goes bankrupt.

This is the kind of result that students jeer at. We made some absurd assumptions and came to an absurd conclusion. But in fact we have built a firm foundation for our next step. We have introduced the concepts of Φ and $\Re_i$, and realized that valuation depends on computing the dynamics of long-run wealth, which in turn depends on our $\Re_i$ relative to Φ and opponents' average $\Re_i$.We will need additional parameters to achieve realistic valuations, in fact to achieve any valuations at all, but our framework is in place. We also know a useful fact, no Monopoly game would ever end with the next feature we are going to work the model.

22 of the 28 properties are organi into eight color groups (two of two pr erties each and six of three properties each). These color groups are usually called "monopolies" but technically they are not monopolies unless one player owns all properties in the grou

Once a player has a monopoly, the rent on all properties in the color group doubles. More important, the player can then buy houses from the Bank. The price for houses is set for each color group, and the rent increases are listed on each property. These rent increases can be large, for example the most expensive and highest rent property Boardwalk has a rent of $50 if owned without its



THE LITERATURE AND STUDY OF THE GAME

Monopoly has spawned a robust literature including curiosities such as accounting workbook Accounting for Monopoly Game Transactions by W. Robert Knechel, Monopoly-based murder mystery Fair Game by Rochelle Krich, math textbook Monopoly Junior Math Madness by Howie Dewin and I. M. Fien (possibly joke names, I'm not sure), investigative thriller about the real invention of the game The Billion Dollar Monopoly® Swindle by Ralph Anspach and business primer Everything I Know About Business I Learned from Monopoly by Alan Axelord. Aside from these, most writing falls into one of two categories. The classic published books—The Monopoly Companion by Philip Orbanes, The Monopoly Book by Maxine Brady and Beyond Boardwalk and Park Place by Noel Gunther and Richard Hutton—offer game tips distilled from experience and musing, along with lots of collateral material, but nothing a finance professor would recognize as an analysis. The other type of work is rarely published (exceptions are Winning Monopoly by Kaz Darzinskis and Robert Ash and Richard Bishop, "Monopoly as a Markov process," Mathematics Magazine 45 (January 1972): 26-29). It consists of extensive mathematical investigation into aspects of the game, but is not brought to the level of useful playing insight.

The situation is analogous to real-life finance before the great modern advances begun by Harry Markowitz, Franco Modigliani and Merton Miller in the 1950s and ending 20 years later with Fischer Black, Robert Merton and Myron Scholes (while this is not an exhaustive list, it's important to add William Sharpe and Gene Fama in the middle). All the mathematical tools had been invented by the time Markowitz published Portfolio Selection, practitioners were using variants of the theories every day, but no one had pulled the math and the practice together in the right way. When people say "ABC really invented the random walk model" or "XYZ figured out option pricing long before Black-Scholes" they miss the point. Modern finance built some tools and did a job. Finding the tool and not using it, or doing the job without the tool, is not the same thing.

I find this point impossible to make to students, because after something is done its difficult to see why it was either hard or important. So I show them how to use these same tools to analyze Monopoly. After you gain true insight and playing advantage from real financial analysis, you see the difference between this and Kaz Darzinskis gathering statistics from ten million simulated games on his home computer or Ash and Bishop figuring out an eigenvector of their 123x123 transition matrix.

color group mate Park Place, $100 if owned with Park Place and $2,000 with five houses (= one hotel) built on it (building houses requires ownership of Park Place). The other six properties consist of four railroads and two utilities. The rents on both of these groups goes up as a player owns more properties within them, but no houses may be built.

If all 28 properties are held as monopolies, that is for each group a single player owns the 2 to 4 properties that make it up, and all monopolies are built to the limit (hotel) then the total rent on the board is $22,790. Dividing by 40 gives a total rent roll of $570 and an average rent roll in a four-person game of $144. Now it is easy for a player to have a rent roll more than $7.60 below the average, so the game gets dangerous.

We can compute the distribution of the difference between a player's rent roll and the average assuming no trading, all players buy all properties from the Bank at the first opportunity and the game has progressed long enough that all properties have been purchased and all resulting monopolies have been developed to the limit. The mean of this distribution is of course zero, and the standard deviation is $28.63. It is not Normal, it is highly skewed to the right. The probability of a player ending up with a rent roll more than $7.60 below the average rent roll is about 40%. It is generally true that some players must trade to survive, which in turn generally means that all players must trade to have hopes of winning. Most home games are won and lost in the trading phase, but good players make trades that are close enough to even that property management skills determine the outcome. Luck has little to do with it.

To account for the option to develop properties, we assume that the rent of a monopoly is a continuous linear function of the amount invested in houses for the color group. Each player's rent roll is now a function of wealth, which is a function of time. We define player i's wealth at time t as $W_i(t)$. Player i's rent roll at time t is $\Re_i(t) = \alpha_i + \beta_i W_i(t)$.

To simplify notation, we are going to consider the value of player 1's properties, and use $\Re(t)$ without the subscript to refer to the sum of the rent rolls of players 2 to n, similarly with α and $W(t)$. β is the combined rent increase per dollar investment for players 2 to n. β's do not add, we will consider later how to combine them in portfolios.

The evolution of $W_1(t)$ is described by

$$\frac{\partial W_1(t)}{\partial t} = \Phi + (n-1)\Re_1(t) - \Re(0).$$

This can be rewritten as:

$$(1)\ \frac{\partial W_1(t)}{\partial t} = \Phi + (n-1)[\alpha_1 + \beta_1 W_1(t)] - [\alpha + \beta W(t)]$$

We know that $\frac{\partial [W_1(t) + W(t)]}{\partial t} = n\Phi$. Solving this gives

$$W_1(t) + W(t) = n\Phi t + W_1(0) + W(0)$$

or $W(t) = n\Phi t + W_1(0) + W(0) - W_1(t)$.

Substituting this into (1) gives

$$\frac{\partial W_1(t)}{\partial t} = \Phi + (n-1)[\alpha_1 + \beta_1 W_1(t)] - [\alpha + \beta(n\Phi t + W_1(0) + W(0) - W_1(t))].$$

The solution, with C as a constant of integration, is

$$W_1(t) = Ce^{[(n-1)\beta_1+\beta]t} + \frac{1}{(n-1)\beta_1+\beta} \times \left[\beta n\Phi\left(t - \frac{1}{(n-1)\beta_1+\beta}\right) - \Phi - (n-1)\alpha_1 - \beta W_1(t) + \Re(0)\right].$$

This is one of the great surprises of Monopoly, the lurking exponential. Without the option to develop properties there is a safe zone of $\frac{\Phi}{n}$ to either side of the mean. If all players stay in the safe zone, the game goes on forever. But now that we have an exponential, every player will either have positive or negative C. Positive C means wealth goes to infinity with time, negative C means it goes to negative infinity (hitting zero along the way and eliminating the player from the game).

This is true of life as well. As Shakespeare wrote in Julius Caesar (before his non-mathematical editor changed it): "There is an exponential tide in the affairs of men, which, taken at the flood, leads on to fortune; omitted, all the polynomial voyage of their life, is bound in shallows and in miseries." Many people have an idea in their head of a safe zone, with linear ups and downs. If Wealth $>$ 0 and Income $>$ Expenses and This Year's Net Income $>$ Last Year's Net Income, things are fine. But every penny you get is an expense item on someone else's income statement. Someone is lying awake at night trying to cut that cost. If you have a valuable asset, skill or franchise, someone smart is working to replace it with something better cheaper. There is no security in great wealth or its derivative great income, or its derivative rapidly increasing income, or any derivative down the line. Exponential trumps growth at any finite power. Things change exponentially.

This is why the criticism that Monopoly teaches ruthless competition is misguided. All players in Monopoly are trying to do is survive. Even a hard-line leftist can't object to that. It is a deep characteristic of creative activity that it produces exponentials. You can only survive by riding on one of these curves. Everyone trying to stand still, or grow linearly, or squarely or cubically; will be left behind. In Monopoly, you end up either rich or bankrupt. In real life, if you choose to participate in creative economic activity, you end up either rich or working for someone rich.

There is a popular misconception about exponentials that infects even quants. It is captured in the term "exponentially fast." "Exponential" does not mean fast, it just means the derivative is proportional to the value. It's a good bet that by the time people notice something is growing fast, it has long since passed its inflection point and is no longer exponential. In fact, it's rare to find anything growing fast in absolute terms that even has a positive second derivative.

It's valuable to spot exponential growth when it is still slow, when most people don't notice. If something went from $1 billion a month ago to $2 billion today, and it's exponential, it will be greater than the total wealth of the world in a year. That doesn't happen often. But if something went from $1 a month ago to $2 today, it might well be exponential. If it stays that way for two years you have a $30 million opportunity and you might be the only one who sees it today. Good Monopoly players know to ignore wealth and income in favor of the exponentially slow factors that will be decisive in the end.

At least one player must have negative C and when she goes bankrupt the parameters change so that at least one other player will be negative. Only one player can survive in Monopoly. To make progress in our analysis, we need to derive an expression for C.

Setting $t = 0$ gives

$$W_1(0) = C + \frac{1}{(n-1)\beta_1+\beta}\left[\frac{-\beta n\Phi}{(n-1)\beta_1+\beta} - \Phi - (n-1)\alpha_1 - W_1(0) + \Re(0)\right].$$

C will be greater than zero if and only if

$$\Re_1(0) + \frac{\beta_1\Phi}{(n-1)\beta_1+\beta} > \frac{\Re(0)}{n-1} + \frac{\beta\Phi}{(n-1)\beta_1+\beta}.$$

This has a natural interpretation. $\Re_1(0)$ is the annuity value of the current rent, $\frac{\beta_1\Phi}{(n-1)\beta_1+\beta}$ is the annuitized value of the option to build. Since each roll of the dice brings Φ dollars into the game from the Bank, $\frac{\beta_1}{(n-1)\beta_1+\beta}$ is the share of that money that belongs to player 1. For consistency

$$\sum_{i=1}^{n} \frac{\beta_i}{(n-1)\beta_i+\beta} = 1,$$

which means that

$$\beta \approx \sqrt[n]{\prod_{i=1}^{n} \beta_i}$$

which is the geometric mean of the individual players' β_i. If $\frac{\beta_1\Phi}{(n-1)\beta_1+\beta}$ is greater than the same quantity for her combined opponents, player 1 has caught the exponential tide. If not, it's the polynomial voyage to shallows and misery.

Since $\Re_1(0) + \frac{\beta_1\Phi}{(n-1)\beta_1+\beta}$ determines the fate of a player, it is natural to evaluate properties according to this measure. It is an annuity value, an amount received every roll of the dice, so we need to divide by an interest rate to convert it to a present value. β is the appropriate market rate of interest per roll of the dice, since $1 of investment in building returns $\$\beta$ per roll of the dice to the average player.

This implies a property valuation of

$$\frac{1}{\beta}\left[\Re_1(0) + \frac{\beta_1\Phi}{(n-1)\beta_1+\beta}\right].$$

Before estimating β_1 and β to get actual numbers for property values, look at the expression itself. It should remind you of two basic financial models, the Gordon Model for stock valuation $\left(Price = \frac{Dividend}{Yield-Growth}\right)$ and the Capital Asset Pricing Model (Excess Security Return = Beta times Excess Market Return). Both of these models figure prominently in introductory finance. Students fail to appreciate them because they are based on assumptions everyone knows are false, and they give results that are useless for practical decision making. Their value is that they give insight into the problem, so you can start in the right direction toward the answer.

Like the Gordon model, we are capitalizing the income stream of the property with an adjustment for growth opportunities, under a simplified and unrealistic assumption about future cash flows. Like the CAPM with systematic and idiosyncratic risk, we are splitting value into two components, $\Re_1(0)$, which adds when properties are combined in a portfolio, and $\beta_1\Phi$ which has a more complex portfolio effect (in our simple model, a player would always choose to develop her highest β_1 property, so only the maximum β_1 in a portfolio matters).

For individual properties, railroads and utilities $\beta_1 = 0$ and $\Re_1$ is the rent divided by 40. For monopolies, we can estimate β_1 as the difference in fully developed and unimproved rent divided by the cost of buying hotels for all properties in the group. This ranges from 3.2% for the green color group (Pacific, North Carolina and Pennsylvania) to 5.5% for the light blue (Oriental, Vermont and Connecticut). The average is about 4%. That is, on the average monopoly, buying \$100 worth of houses adds about \$160 to total rent, which when divided by 40 means a \$4 increase in $\Re_1$.

If we set $\beta = 4\%$ as suggested above, the value of properties that cannot be developed is 0.625 times the rent, regardless of the number of players. The value for a color group is 0.625 times the rent plus, in a four person game, $25 \times \frac{\$30.40\times\beta_1}{3\times\beta_1+.04}$. The table on this page gives these values.

Experienced Monopoly players will find these values absurd. They are much too low, the railroads are overvalued compared to the monopolies and the monopolies are too close in value. Although we made a lot of simplifying assumptions that are false, two in particular are causing the misvaluation.

Before discussing those, I want to point out that casual players overvalue monopolies, especially the expensive ones, and undervalue properties that cannot be developed (single properties, railroads and utilities). As mentioned above, players change the rules to inject more money in the game. That increases property values both directly, by increasing Φ which appears in the valuation formula for monopolies (but not properties that cannot be developed), and indirectly by forcing down interest rates. The reduction in interest rates reduces the advantage of high β_1 monopolies (like the light blue) over low β_1 monopolies (like the green).

The two major problem assumptions in our model are continuous linear property development and a single, constant interest rate. Early in the game the light blue monopoly's 5.5% β_1 allows quick development, and often a quick victory. But if the game continues, money builds and the green monopoly's total rent with hotels of \$3,950 (more than twice the \$1,700 of Oriental/Vermont/Connecticut) assumes greater importance, even though it offers only a 3.2% return on investment. But our model gives no credit to that second effect because it assumes there is no limit to development. This is a trivial problem, however. We can easily modify our equations for discrete development functions. We don't get neat formulae that provide insight, but we do get more realistic practical valuations.

The assumption of constant interest rates is both quantitatively more important and theoretically more difficult to deal with. Interest rates start out very low in Monopoly, under 1%. No one goes bankrupt at low interest rates so eventually people will trade and develop until interest rates are high, often much higher than 4%. These rates are unstable and either the development continues until players start running up against maximum development limits, or so much money is destroyed that players get thrown back into a low-development, low-interest rate game. In order to value properties we will have to account for different types of interest rates, their term structure and their evolution. This is going to require a full-fledged interest rate model, as complex as those used to price exotic fixed-income derivatives.

To see the size of the interest rate effect, consider that under our simple assumptions it takes about $40 \times [\log(28.5) - \log(0.5)] \approx 161$ dice rolls for all the properties to be purchased. Typically there is little or no development until around this time in the game. A monopoly worth \$10,000 when development commences has a present value, at 4% per dice roll, of only \$18 when play begins. We haven't understated values by this much, because interest rates are never zero in Monopoly, but it is an important effect. To handle it, we'll have to add a random term to our differential equation.

In Part II we will also tackle the following problems:

- Adjust all formulae for the unequal probabilities of visiting each square. The biggest effect will be that Φ will fall from \$30.40 to about \$23 due to time spent in Jail reducing the frequency of passing Go.
- Consider the effect of property mortgaging and selling houses at half price.
- Solve the risk management equation for the optimal liquidity policy
- Compute the portfolio effects both micro (valuing individual properties) and macro (valuing combinations of monopolies).
- Consider the effect of building shortages (you can consider yourself a good Monopoly player if know how to bid for houses and hotels).

TRIVIAL PURSUITS (TO GO ON THIRD SPREAD)

200 million sets have been sold in 26 languages and Hasbro prints twice as much Monopoly money every year as the US Mint prints real money—

The more genteel UK rules don't mention bankruptcy, instead: "The idea of the game is to buy and rent or sell properties so profitably that players increase their wealth—the wealthiest becoming the eventual winner." The rules go on to explain the reason for accumulating money and suggest that it is "wise" to do so.

Monopoly is responsible for the near-universal misconception that a deed is a document that proves ownership, in fact a deed is a document that transfers ownership.

• Adjust for cards that require assessments for houses and hotels and other payments such as to get out of Jail, payments to other players required by cards and the 10% income tax option.

Until then, I encourage you to take a crack at these problems yourself, or develop your own Monopoly pricing theory. If you're a practitioner, buy a set (or dust one off from the closet, or get one of the popular shareware computer versions) and put your quant skills to the test.